INEQUIVALENT SURFACE-KNOTS WITH THE SAME KNOT QUANDLE

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Abstract. We have a knot quandle and a fundamental class as invariants for a surface-knot. These invariants can be defined for a classical knot in a similar way, and it is known that the pair of them is a complete invariant for classical knots. In this paper, we compare a situation in surface-knot theory with that in classical knot theory, and prove the following: There exist arbitrarily many inequivalent surface-knots of genus g with the same knot quandle, and there exist two inequivalent surface-knots of genus g with the same knot quandle and with the same fundamental class.

1. Introduction

We consider a *knot quandle* [15, 18], $Q(F)$, and a *fundamental class* [5] (cf. [26]), $[F] \in H_3^{\mathrm{Q}}(Q(F))$, as invariants of a surface-knot F, where a surface-knot means an oriented closed connected surface embedded in $\mathbb{R}^4$. The fundamental class can be considered as a universal object concerning to a *quandle cocycle invariant* (See Section 2.5). When the invariants are given, what we want to know might be the following:

- What kind of information can be extracted from them?
- How powerful are they?

For the first question, it is known in [15, 18] that the knot quandle of a surface-knot F can recover information of the knot group $\pi_1(\mathbb{R}^4 \setminus F)$, for example. There are some relation of the knot quandle to the braid index [25], to the unknotting number [14] and to the sheet number [21]. There are also some relation of the fundamental class to the non-invertibility [3, 1, 13], to the triple point number [22, 23, 11, 26], to the triple point cancelling number [14], and to the ribbon concordance [7].

For the second question, it is known in [2] that the knot quandle can distinguish all elements of a class of twist-spun S^2-knots obtained from torus knots, for example. In this paper, we focus on the second question and compare a situation in surface-knot theory with that in classical knot theory.

1.1. **The case of classical knots.** Similarly, we have a knot quandle $Q(k)$ and a fundamental class $[k] \in H_2^{\mathrm{Q}}(Q(k))$ as invariants of a classical knot k (cf. [8]). For a classical knot k, let $-k$ denote the classical knot obtained from k by reversing the orientation, and k^* denote the mirror image of k. Then the following three facts are known.

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- Fact (cf. [4, Proof of Theorem 9.1]): For a classical knot k, there exists a canonical quandle isomorphism $\phi : Q(k) \to Q(-k^*)$ such that the induced homomorphism $\phi_* : H_2^{\mathrm{Q}}(Q(k)) \to H_2^{\mathrm{Q}}(Q(-k^*))$ satisfies the condition $\phi_*[k] = -[-k^*]$.
- Theorem due to Joyce [15] and Matveev [18]: For classical knots k and k', if there exists a quandle isomorphism $\phi : Q(k) \to Q(k')$, then k is equivalent to k' or $-(k')^*$.
- Theorem due to Eisermann [8]: For classical knots k and k', if there exists a quandle isomorphism $\phi : Q(k) \to Q(k')$ such that the induced homomorphism ϕ_* satisfies the condition $\phi_*[k] = [k']$, then k is equivalent to k'.

Roughly speaking, Joyce–Matveev's theorem says that the knot quandle is an almost complete invariant for classical knots, and Eisermann's theorem says that the pair of the knot quandle and the fundamental class is a complete invariant for them.

Remark 1.1. Eisermann [8] also proved:

- For a trivial classical knot k, we have $H_2^{\mathrm{Q}}(Q(k)) \cong 0$.
- For a non-trivial classical knot k, we have $H_2^{\mathrm{Q}}(Q(k)) \cong \mathbb{Z}$ and the fundamental class $[k]$ is a generator.

On the other hand, as far as the author knows, there is not so much result about the structure of $H_3^{\mathrm{Q}}(Q(F))$ for a surface-knot F.

1.2. **Problem setting.** For a surface-knot F, let $-F$ denote the surface-knot obtained from F by reversing the orientation, and F^* denote the mirror image of F. It is known that the assertion corresponding to the first fact in Section 1.1 also holds for a surface-knot F, that is, there exists a canonical quandle isomorphism $\phi : Q(F) \to Q(-F^*)$ such that the induced homomorphism $\phi_* : H_3^{\mathrm{Q}}(Q(F)) \to H_3^{\mathrm{Q}}(Q(-F^*))$ satisfies the condition $\phi_*[F] = -[-F^*]$ (cf. [4, Proof of Theorem 9.2]). Then we consider the following problem.

Problem 1.2.

(I) Does the assertion corresponding to Joyce–Matveev's theorem hold for surface-knots?

(II) Does the assertion corresponding to Eisermann's theorem hold for surface-knots?

Since the knot quandle does not have information of the genus of a surface-knot, we fix a non-negative integer g and consider the above problem for surface-knots of genus g. To make the problem concrete, we consider the following five conditions for two surface-knots, F and F', of genus g:

(i) There exists a quandle isomorphism $\phi : Q(F) \to Q(F')$.

(ii) There exists a quandle isomorphism $\phi : Q(F) \to Q(F')$ such that
$$\phi_*[F] = [F'] \in H_3^{\mathrm{Q}}(Q(F')).$$

(ii') There exists a quandle isomorphism $\phi : Q(F) \to Q(F')$ such that
$$\phi_*[F] = \pm[F'] \in H_3^{\mathrm{Q}}(Q(F')).$$

(iii) The surface-knot F is equivalent to F'.

(iii') The surface-knot F is equivalent to F' or $-(F')^*$.

By definition, we have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (ii'), and (iii) $\Rightarrow$ (iii'). As mentioned above, we also have (iii') $\Rightarrow$ (ii') $\Rightarrow$ (i). Then we can reformurate Problem 1.2 as follows:

Problem 1.3. (Reformultation of Problem 1.2)

(I) Does the condition (i) imply the condition (iii')?
(II) Does the condition (ii) imply the condition (iii)?

Moreover, by the fact that (iii') $\Rightarrow$ (ii') $\Rightarrow$ (i), we can divide (I) into two parts.

(I_1) Does the condition (i) imply the condition (ii')?
(I_2) Does the condition (ii') imply the condition (iii')?

The main result of this paper is to give negative answers to Problem 1.3.

Theorem 1.4. *For a non-negative integer g, there exist arbitrarily many surface-knots of genus g such that any two of them satisfy the condition* (i) *but do not satisfy the condition* (ii')*.*

Theorem 1.5. *For a non-negative integer g, there exist two surface-knots of genus g such that they satisfy the condition* (ii) *but do not satisfy the condition* (iii')*. Moreover, infinitely many such pairs exist.*

Theorem 1.4 gives a negative answer to Problem 1.3 (I_1), and Theorem 1.5 gives a negative answer to Problem 1.3 (I_2) and (II).

Remark 1.6. It follows from Theorem 1.4 that there exist arbitrarily many inequivalent surface-knots of genus g with the same knot group. We note that the more stronger assertion is known for surface-knots of genus zero: There exist infinitely many S^2-knot with the same knot group [24].

The rest of this paper is organized as follows. In Section 2, we review the basic definitions including knot quandles and fundamental classes of surface-knots, and give Lemma 2.1 and Corollary 2.4 which are keys to proving theorems. Section 3 and Section 4 are devoted to proving Theorem 1.4 and Theorem 1.5 respectively.

2. Definitions and Lemmas

2.1. **Surface-knots and diagrams.** A *surface-knot* is a closed connected oriented surface embedded locally flatly in $\mathbb{R}^4$ (or in the 4-sphere S^4). Two surface-knots are said to be *equivalent* if they are related by an ambient isotopy of $\mathbb{R}^4$. For a fixed projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$, by perturbing a surface-link F if necessary, we may assume that the projection $\pi|_F$ is *generic*, that is, $\pi|_F$ has double points, isolated triple points and isolated branch points in the image as its singularities. A *diagram* of a surface-knot is a generic projection image equipped with height information, where one of two sheets along each double point curves is broken depending on the relative height. A diagram consists of a collection of sheets, and is regarded as a compact oriented surface in $\mathbb{R}^3$. We refer to [6] for more details.

2.2. **Quandles and knot quandles.** A *quandle* [15, 18], X, is a non-empty set with a binary operation $(a, b) \to a * b$ satisfying the following axioms.

(Q1) For any $a \in X$, $a * a = a$.
(Q2) For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
(Q3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A function $f : X \to Y$ between quandles is a *homomorphism* if $f(a*b) = f(a)*f(b)$ for any $a, b \in X$.

Let D be a diagram of a surface-link F, and let $E = \{s_1, \dots, s_m\}$ be the set of all sheets of D. Using the orientation of F and that of $\mathbb{R}^3$, we give a normal vector to each sheet. The *knot quandle* [15, 18], $Q(F)$, of F is a quandle generated by $E = \{s_1, \dots, s_m\}$ with the following defining relations. Along a double point curve, let s_j be the over-sheet and s_i (resp. s_k) the under-sheet which is behind (resp. in front of) the over-sheet s_j with respect to the normal vector of s_j. The defining relation is given by $s_i * s_j = s_k$ along the double point curve. We note that $Q(F)$ is independent of the choice of the diagram of F. The following lemma will be used to construct surface-knots satisfying the condition (i).

Lemma 2.1. *For surface-knots F_0 and F, consider the connected sums $F_0\#F$ and $F_0\# - F^*$. Then $Q(F_0\#F)$ has the same presentation as $Q(F_0\# - F^*)$. In particular, $Q(F_0\#F)$ is isomorphic to $Q(F_0\# - F^*)$.*

Proof. A presentation of $Q(F_0\#F)$ can be obtained from that of $Q(F_0)$ and that of $Q(F)$ by adding a relation such as $a_0 = a$, where a_0 (resp. a) is a generator of $Q(F_0)$ (resp. $Q(F)$) corresponding to a sheet of a diagram of F_0 (resp. F). Since $Q(F)$ has the same presentation as $Q(-F^*)$, the result follows. □

Remark 2.2. The above lemma does not hold for classical knots in general. Take the right-handed trefoils as k_0 and k for example. Then it is known in [20, p.220] that the granny knot is not equivalent to the square knot up to orientation. (See Remark 3.2 for an alternative proof of this fact.) Thus we have that $Q(k_0\#k)$ is not isomorphic to $Q(k_0\# - k^*)$.

2.3. **Quandle homology theory.** Before defining the fundamental class, we briefly review the quandle homology theory defined in [3]. For $n > 0$, let $C_n^{\mathrm{R}}(X)$ be the free abelian group generated by n-tuples $(x_1, x_2, \dots, x_n)$ of elements of a quandle X. Put $C_n^{\mathrm{R}}(X) = 0$ for $n \le 0$. We define the boundary map $\partial_n : C_n^{\mathrm{R}}(X) \to C_{n-1}^{\mathrm{R}}(X)$ by

$$\begin{aligned}\partial_n(x_1, \dots, x_n) \quad = \quad (-1)^{n-1} \textstyle\sum_{i=1}^{n} (-1)^i \quad & \big\{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ & -(x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)\big\}\end{aligned}$$

for $n > 1$, and $\partial_n = 0$ for $n \le 1$. It is easily verified that $C_*^{\mathrm{R}}(X) = (C_n^{\mathrm{R}}(X), \partial_n)$ is a chain complex.

For $n > 1$, let $C_n^{\mathrm{D}}(X)$ be the submodule of $C_n^{\mathrm{R}}(X)$ generated by n-tuples $(x_1, x_2, \dots, x_n)$ with $x_i = x_{i+1}$ for some i ($i = 1, 2, \dots, n-1$). Put $C_n^{\mathrm{D}}(X) = 0$ for $n \le 1$. Quandle axiom (Q1) ensures that $\partial_n(C_n^{\mathrm{D}}(X)) \subset C_{n-1}^{\mathrm{D}}(X)$, hence $C_*^{\mathrm{D}}(X) = (C_n^{\mathrm{D}}(X), \partial_n)$ is a subcomplex of $C_*^{\mathrm{R}}(X)$.

Put $C_n^{\mathrm{Q}}(X) = C_n^{\mathrm{R}}(X)/C_n^{\mathrm{D}}(X)$ and $C_*^{\mathrm{Q}}(X) = (C_n^{\mathrm{Q}}(X), \partial_n)$, where all the induced boundary operators are again denoted by ∂_n. For an element x of $C_n^{\mathrm{R}}(X)$, we denote the equivalence class of x by $x|_{\mathrm{Q}} \in C_n^{\mathrm{Q}}(X)$. The nth groups of cycles and boundaries of $C_*^{\mathrm{Q}}(X)$ are denoted by $Z_n^{\mathrm{Q}}(X)$ and $B_n^{\mathrm{Q}}(X)$, and the nth homology group of this complex is called the nth *quandle homology group* [3] and is denoted by $H_n^{\mathrm{Q}}(X)$. For an abelian group A, define the cochain complex

$$C^*_{\mathrm{W}}(X; A) = \mathrm{Hom}_{\mathbb{Z}}(C_*^{\mathrm{W}}(X), A), \quad \delta^* = \mathrm{Hom}(\partial_*, \mathrm{id})$$

in the usual way, where W = R, D or Q. The nth groups of cocycles and coboundaries of $C^*_{\mathrm{Q}}(X; A)$ are denoted by $Z^n_{\mathrm{Q}}(X; A)$ and $B^n_{\mathrm{Q}}(X; A)$, and the nth cohomology

group of this complex is called the nth *quandle cohomology group* [3] and is denoted by $H^n_{\mathrm{Q}}(X;A)$.

2.4. **Fundamental classes.** Let D be a diagram of a surface-link F and let $E = \{s_1,\dots,s_m\}$ be the set of the sheets of D. We often regard an element of E as the element of the knot quandle $Q(F)$.

At a triple point r of D, let $\vec{v_t}$, $\vec{v_m}$ and $\vec{v_b}$ be the normal vectors to the top, middle, and bottom sheet respectively. For the triple point r, the *sign* $\varepsilon(r)$ is 1 if the ordered triple $(\vec{v_t},\vec{v_m},\vec{v_b})$ matches the orientation of $\mathbb{R}^3$, and -1 otherwise.

For a triple point r of D, $C(r) = (s_b, s_m, s_t)$ is a triplet of elements of $Q(F)$, where s_b is one of the four bottom sheets from which the normal vectors of the middle and top sheets point, s_m is one of the two middle sheets from which the normal vector of the top sheet points, and s_t is the top sheet.

For a triple point r, the *Boltzmann weight* $B(r) \in C^{\mathrm{R}}_3(Q(F))$ is defined by

$$B(r) := \varepsilon(r)C(r)\ \Big(= \pm(s_b, s_m, s_t)\Big).$$

Let $|D| \in C^{\mathrm{R}}_3(Q(F))$ be the sum of the Boltzmann weights $B(r)$ of all triple points of the diagram D. Then we have the following (cf. [3, Theorem 5.6]):

- $|D|\big|_{\mathrm{Q}} \in Z^{\mathrm{Q}}_3(Q(F))$, and
- $|D'|\big|_{\mathrm{Q}} - |D|\big|_{\mathrm{Q}} \in B^{\mathrm{Q}}_3(Q(F))$, for any other diagram D' of F.

Thus the homology class of $|D|\big|_{\mathrm{Q}}$ is independent of the choice of the diagram D, and the *fundamental class* [5] (cf. [26]), $[F]$, of a surface-link F is defined by

$$[F] := \Big[|D|\big|_{\mathrm{Q}}\Big] \in H^{\mathrm{Q}}_3(Q(F)).$$

2.5. **Quandle cocycle invariants.** Although a quandle cocycle invariant [3] was originally introduced as an invariant for a surface-knot, we use it as a tool for distinguishing given fundamental classes (See Lemma 2.3 and Corollary 2.4 below).

Let F be a surface-link and let $[F] \in H^{\mathrm{Q}}_3(Q(F))$ be the fundamental class of F. For a finite quandle X, a abelian group A and a 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(X;A)$, we define a *quandle cocycle invariant* [3], $\Phi_\theta(F)$, by

$$\Phi_\theta(F) = \sum_{c:Q(F)\to X} \langle\, c_*([F]), [\theta]\,\rangle \ \in \mathbb{Z}[A],$$

where $c_* : H^{\mathrm{Q}}_3(Q(F)) \to H^{\mathrm{Q}}_3(X)$ is a map induced from a quandle homomorphism $c : Q(F) \to X$, the element $[\theta]$ is a cohomology class of θ, and

$$\langle\ ,\ \rangle : H^{\mathrm{Q}}_3(X) \underset{\mathbb{Z}}{\otimes} H^3_{\mathrm{Q}}(X;A) \to A$$

is a Kronecker product. We note that the above summation is finite, since the cardinarity of X is finite. The following are easy consequences of the construction of quandle cocycle invariants, and Corollary 2.4 plays an important role in the proof of Theorem 1.4.

Lemma 2.3. *For surface-knots F and F', if there exists a quandle isomorphism $f : Q(F) \to Q(F')$ such that $f_*[F] = [F']$, then we have $\Phi_\theta(F) = \Phi_\theta(F')$ for any finite quandle X, any abelian group A and any 3-cocycle θ of $Z^{\mathrm{Q}}_3(X;A)$.*

Corollary 2.4. *For surface-knots F and F', if there exists a finite quandle X, an abelian group A and a 3-cocycle θ of $Z^3_{\mathrm{Q}}(X;A)$ such that*

$$\Phi_\theta(F) \neq \Phi_\theta(F') \text{ and } \Phi_\theta(F) \neq \Phi_\theta(-(F')^*),$$

then F and F' do not satisfy the condition (ii').

3. Proof of Theorem 1.4

Before proving Theorem 1.4, we define two S^2-knots $F_{p,1}$ and $F_{p,2}$, and study their properties. For an odd prime integer p, let K_p be the 2-twist spun S^2-knot obtained from a $(2,p)$-torus knot. Let $F_{p,1}$ be the connected sum of two copies of K_p, and $F_{p,2}$ be the connected sum of K_p and $-(K_p)^*$.

For a surface-knot F, let $\Phi_p(F)$ denote the quandle cocycle invariant of F associated with Mochizuki's 3-cocycle [19], $\theta_p \in Z^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p)$, of the dihedral quandle R_p and the coefficient group $\mathbb{Z}_p$. We note that the invariant $\Phi_p(F)$ takes values in $\mathbb{Z}[t,t^{-1}]/(t^p-1)$ ($\cong \mathbb{Z}[\mathbb{Z}_p]$). Using Asami and Satoh's computation [1], we have the following:

$$\Phi_p(F_{p,1}) = p\left(\sum_{k=0}^{p-1} t^{2k^2}\right)^2 \quad \text{and} \quad \Phi_p(F_{p,2}) = p\left(\sum_{k=0}^{p-1} t^{2k^2}\right)\left(\sum_{k=0}^{p-1} t^{-2k^2}\right).$$

Proposition 3.1. *If p is an odd prime integer with $p \equiv 3 \pmod 4$, then $\Phi_p(F_{p,1})$ is not equal to $\Phi_p(F_{p,2})$ in $\mathbb{Z}[t,t^{-1}]/(t^p-1)$.*

Proof. To compare thier values in $\mathbb{Z}[t,t^{-1}]/(t^p-1)$, it is sufficient to calculate "constant terms" of them, where the constant term of $\sum_i a_i t^i$ is defined to be

$$\sum_{i\equiv 0 \pmod p} a_i \quad \in \mathbb{Z}.$$

For integers $i,j \in \{0,\dots,p-1\}$, it follows from the condition $p \equiv 3 \pmod 4$ that $2(i^2+j^2) \equiv 0 \pmod p$ if and only if $(i,j)=(0,0)$. Hence the constant term of $\Phi_p(F_{p,1})$ in $\mathbb{Z}[t,t^{-1}]/(t^p-1)$ is equal to p.

For integers $i,j \in \{0,\dots,p-1\}$, it is easy to see that $2(i^2-j^2) \equiv 0 \pmod p$ if and only if

$$\begin{aligned}(i,j) \quad = \quad & (0,0),(1,1),\dots,(p-1,p-1),\\ & (1,p-1),(2,p-2),\dots,(p-1,1).\end{aligned}$$

Hence the constant term of $\Phi_p(F_{p,2})$ in $\mathbb{Z}[t,t^{-1}]/(t^p-1)$ is equal to $p(2p-1)$. □

Proof of Theorem 1.4. We construct S^2-knots satisfying the condition of Theorem 1.4. Let $\mathcal{P}$ be the set of odd prime integers with $p \equiv 3 \pmod 4$, and take a subset $\{p_1,\dots,p_n\}$ of $\mathcal{P}$ for any non-negative integer n. We notice that the cardinality of $\mathcal{P}$ is countable. Given an n-tuple $I=(e_1,\dots,e_n) \in \{1,2\}^n$, we consider the S^2-knot

$$F_I = F_{p_1,e_1}\#\dots\#F_{p_n,e_n},$$

and claim that these 2^n surface-knots satisfy the condition. For any two distinct elements $I=(e_1,\dots,e_n)$ and $I'=(e'_1,\dots,e'_n)$ of $\{1,2\}^n$, we have $Q(F_I)\cong Q(F_{I'})$ by Lemma 2.1, that is, F_I and $F_{I'}$ satisfy the condition (i). Since $I \neq I'$, there exists some j ($j=1,\dots,n$) such that $e_j \neq {e'}_j$. Thus we have

$$\Phi_{p_j}(F_I) = \Phi_{p_j}(F_{p_j,e_j}) \neq \Phi_{p_j}(F_{p_j,{e'}_j}) = \Phi_{p_j}(F_{I'})$$

by Proposition 3.1. We can also show

$$\Phi_{p_j}(F_I) \neq \Phi_{p_j}(-(F_{I'})^*)$$

in a similar way. Hence F_I and $F_{I'}$ do not satisfy the condition (ii') by Corollary 2.4.

When the genus g is greater than zero, we consider the connected sum of F_I and a trivial surface-knot of genus g. Then these 2^n surface-knots of genus g satisfy the condition of Theorem 1.4. □

Remark 3.2. We give an alternative proof of the fact mentioned in Remark 2.2. By the above proof, $F_{3,1}$ $(= K_3 \# K_3)$ is not equivalent to $F_{3,2}$ $(= K_3 \# -(K_3)^*)$. Then, for the right-handed trefoil knot (i.e., $(2,3)$-torus knot) k_3, it follows from [17] that $k_3 \# k_3$ is not equivalent to $k_3 \# -(k_3)^*$. Since the trefoil knot is invertible, the granny knot, $k_3 \# k_3$, is not equivalent to the square knot, $k_3 \# (k_3)^*$, up to orientation.

4. Proof of Theorem 1.5

The proof is divided into two cases: One is the case where $g = 0$ and the other is the case where $g > 0$.

4.1. $g = 0$ **case.** Take integers $n, p, q > 5$ such that p and q are relatively prime. Let K be a n-twist spun S^2-knot obtained from a (p,q)-torus knot, and $\widehat{K}$ be an S^2-knot obtained from K by Gluck surgery [9]. We remark that the exterior $E(K)$ of the S^2-knot K is homeomorphic to the exterior $E(\widehat{K})$ of $\widehat{K}$. It is known in [10] that the ambient space of $\widehat{K}$ is homeomorphic to the 4-sphere S^4 and that $\widehat{K}$ is not equivalent to K up to orientation. In particular, K and $\widehat{K}$ does not satisfy the condition (iii').

Let Σ be the trivial surface-knot of genus two, and consider the two surface-knots $K \# \Sigma$ and $\widehat{K} \# \Sigma$. We notice that the exterior $E(K \# \Sigma)$ is homeomorphic to $E(\widehat{K} \# \Sigma)$. Then $\widehat{K} \# \Sigma$ is equivalent to $K \# \Sigma$, since a surface-knot of genus greater than one is determined by its exterior [12]. Hence we have

$$Q(K) \xrightarrow[\cong]{\phi_1} Q(K \# \Sigma) \xrightarrow[\cong]{\phi_2} Q(\widehat{K} \# \Sigma) \xrightarrow[\cong]{\phi_3} Q(\widehat{K})$$

and

$$(\phi_3 \circ \phi_2 \circ \phi_1)_*[K] = (\phi_3 \circ \phi_2)_*[K \# \Sigma] = (\phi_3)_*[\widehat{K} \# \Sigma] = [\widehat{K}],$$

where the map ϕ_1 (resp. ϕ_3) is induced by doing the connected sum of the trivial surface-knot Σ to K (resp. $\widehat{K}$), and the map ϕ_2 is induced from the equivalence between $K \# \Sigma$ and $\widehat{K} \# \Sigma$. When we vary integers n, p and q, we can obtain infinitely many such pairs.

4.2. $g > 0$ **case.** Let $T(k)$ denote the spun T^2-knot obtained from a non-trivial classical knot k, and let $\tilde{T}(k)$ denote the turned spun T^2-knot obtained from k. Take a ribbon surface-knot G of genus $g - 1$ (≥ 0) and consider the two surface-knots $G \# T(k)$ and $G \# \tilde{T}(k)$ of genus g. It is easy to see that these two surface-knots satisfy the condition (ii). We note that the fundamental classes of them are equal to zero elements.

To distinguish them, we use Kawauchi's Gauss sum invariant [16, p.1047], $\varsigma(F) \in \mathbb{Z}$, of a surface-knot F. It is known in [16] that $\varsigma(G) = 2^{g-1}$, $\varsigma(T(k)) = 2$ and $\varsigma(\tilde{T}(k)) = 0$. Using the connected sum formula [16, Theorem 1.2]

$$\varsigma(F_1 \# F_2) = \varsigma(F_1)\varsigma(F_2),$$

we have

$$\varsigma(G \# T(k)) = 2^g \neq 0 = \varsigma(G \# \tilde{T}(k)),$$

and it follows that they do not satisfy the condition (iii'). When we vary a non-trivial classical knot k, we can obtain infinitely many such pairs.

Remark 4.1. We may take any surface-knot G of genus $g-1$ as long as it satisfies the condition $\varsigma(G) \neq 0$, though we take a ribbon surface-knot as G in the above proof.

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